

THE BEHAVIOR OF THE MAXIMAL DEGREE OF THE KHOVANOV HOMOLOGY UNDER TWISTING

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ABSTRACT. In this paper, we study the behavior of the maximal homological degree of the non-zero Khovanov homology groups under twisting.

1. INTRODUCTION

In [3], for each oriented link L , Khovanov defined a graded chain complex whose graded Euler characteristic is equal to the Jones polynomial of L . Its homology groups are link invariants and called the Khovanov homology groups. Throughout this paper, we only consider the rational Khovanov homology. The Khovanov homology has two gradings, homological degree i and q -grading j . By $\mathrm{KH}^i(L)$, we denote the homological degree i term of the Khovanov homology groups of a link L and by $\mathrm{KH}^{i,j}(L)$, we denote the homological degree i and q -grading j term of the Khovanov homology groups of L .

The maximal homological degree of the non-zero Khovanov homology groups of a link gives a lower bound of the minimal positive crossing number of the link (see Proposition 2.2). The minimal positive crossing number of a link is the minimal number of the positive crossings of diagrams of the link. From this fact, it seems that the Khovanov homology estimates the positivity of links.

Stošić [4, Theorem 2] showed that the maximal homological degree of the non-zero Khovanov homology groups of the $(2k, 2kn)$ -torus link is $2k^2n$. By using the same method as Stošić's, the author [5, Corollary 1.2] proved that the maximal homological degree of the non-zero Khovanov homology groups of the $(2k+1, (2k+1)n)$ -torus link is $2k(k+1)n$. These results intimate that the maximal degree of the non-zero Khovanov homology groups grows as the number of full-twists grows.

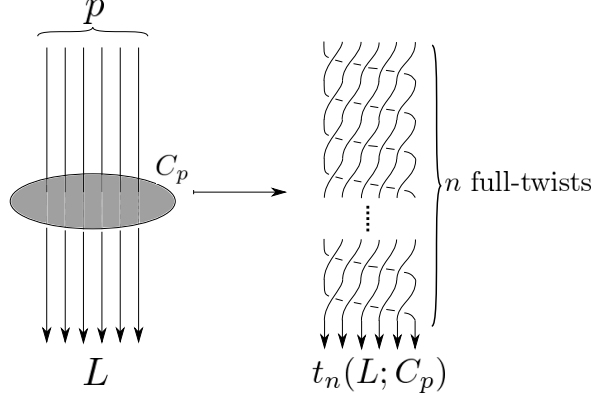
Let L be an oriented link and C_p be a disk which intersects with L at p points transversely with the same orientations as in Figure 1. Then, for any positive integer n , we define a link $t_n(L; C_p)$ as the link obtained from L by adding n full-twists at C_p .

We can consider the following question.

Question 1.1. *Let L and C_p be as above. Then, does the following hold?*

$$\lim_{n \rightarrow \infty} \frac{\max\{i \in \mathbf{Z} \mid \mathrm{KH}^i(t_n(L; C_p)) \neq 0\}}{n} = \lim_{n \rightarrow \infty} \frac{\max\{i \in \mathbf{Z} \mid \mathrm{KH}^i(T_{p,pn}) \neq 0\}}{n}$$

where $T_{p,pn}$ is the positive (p, pn) -torus link.

FIGURE 1. $t_n(L; C_p)$.

Note that Stošić and the author proved

$$\lim_{n \rightarrow \infty} \frac{\max\{i \in \mathbf{Z} \mid \text{KH}^i(T_{p,pn}) \neq 0\}}{n} = \begin{cases} 2k^2 & \text{if } p = 2k, \\ 2k(k+1) & \text{if } p = 2k+1. \end{cases}$$

In [5], the author proved the following which is an evidence that the equality in Question 1.1 holds.

Theorem 1.2 ([5, Theorem 1.3 and Proposition 1.4]). *Let K be an oriented knot. Denote the (p, pn) -cabling of K by $K(p, pn)$ for positive integers p and n . Assume that each component of $K(p, pn)$ has an orientation induced by K , that is, each component of $K(p, pn)$ is homologous to K in the tubular neighborhood of K . Put $c_+(K) := \min\{c_+(D) \mid D \text{ is a diagram of } K\}$, where $c_+(D)$ is the number of the positive crossings of D . If $n \geq 2c_+(K)$, then we have the following for any positive integer k .*

$$\begin{aligned} \max\{i \in \mathbf{Z} \mid \text{KH}^i(K(2k, 2kn)) \neq 0\} &= 2k^2n, \\ 2k(k+1)n \leq \max\{i \in \mathbf{Z} \mid \text{KH}^i(K(2k+1, (2k+1)n)) \neq 0\} &\leq 2k(k+1)n + c_+(K). \end{aligned}$$

In particular, we have

$$\lim_{n \rightarrow \infty} \frac{\max\{i \in \mathbf{Z} \mid \text{KH}^i(K(p, pn)) \neq 0\}}{n} = \begin{cases} 2k^2 & \text{if } p = 2k, \\ 2k(k+1) & \text{if } p = 2k+1. \end{cases}$$

In this paper, we consider the Question 1.1 for $p = 2$. Precisely we prove the following.

Theorem 1.3 (Main Theorem). *Let L be an oriented link and C be a disk which intersects with L at two points with the same orientations as in Figure 2. Then we have*

$$\lim_{n \rightarrow \infty} \frac{\max\{i \in \mathbf{Z} \mid \text{KH}^i(t_{n/2}(L; C)) \neq 0\}}{n} = 1,$$

where $t_{n/2}(L; C)$ is the link obtained from L by adding n half-twists at C .

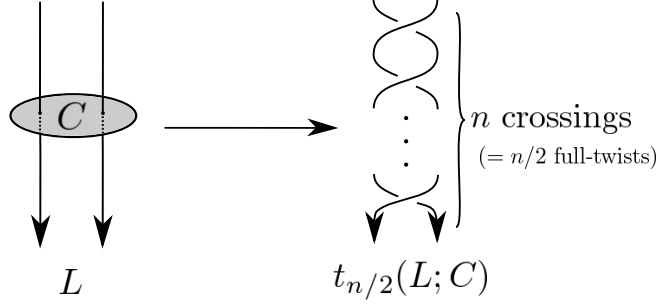


FIGURE 2. Definition of the link $t_{n/2}(L; C)$. The integer n is the number of added half-twists.

This paper is organized as follows: In Section 2, we recall the definition of the Khovanov homology. In Section 3, we prove our main theorem (Theorem 1.3).

2. KHOVANOV HOMOLOGY

2.1. The definition of Khovanov homology. In this subsection, we recall the definition of the (rational) Khovanov homology. Let L be an oriented link. Take a diagram D of L and an ordering of the crossings of D . For each crossing of D , we define 0-smoothing and 1-smoothing as in Figure 3. A smoothing of D is a diagram where each crossing of D is changed by either 0-smoothing or 1-smoothing. Let n

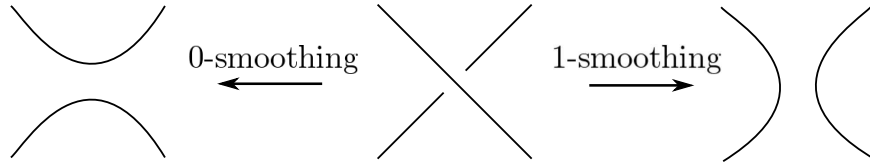


FIGURE 3. 0-smoothing and 1-smoothing.

be the number of the crossings of D . Then D has 2^n smoothings. By using the given ordering of the crossings of D , we have a natural bijection between the set of smoothings of D and the set $\{0, 1\}^n$, where, to any $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{0, 1\}^n$, we associate the smoothing D_ε where the i -th crossing of D is ε_i -smoothed. Each smoothing D_ε is a collection of disjoint circles.

Let V be a graded free \mathbf{Q} -module generated by 1 and X with $\deg(1) = 1$ and $\deg(X) = -1$. Let k_ε be the number of the circles of the smoothing D_ε . Put $M_\varepsilon = V^{\otimes k_\varepsilon}$. The module M_ε has a graded module structure, that is, for $v = v_1 \otimes \dots \otimes v_{k_\varepsilon} \in M_\varepsilon$, $\deg(v) := \deg(v_1) + \dots + \deg(v_{k_\varepsilon})$. Then define

$$C^i(D) := \bigoplus_{|\varepsilon|=i} M_\varepsilon\{i\},$$

where $|\varepsilon| = \sum_{i=1}^n \varepsilon_i$. Here, $M_\varepsilon\{i\}$ denotes M_ε with its gradings shift by i (for a graded module $M = \bigoplus_{j \in \mathbf{Z}} M^j$ and an integer i , we define the graded module $M\{i\} = \bigoplus_{j \in \mathbf{Z}} M^j\{i\}$ by $M\{i\}^j = M^{j-i}$).

The differential map $d^i: C^i(D) \rightarrow C^{i+1}(D)$ is defined as follows. Fix an ordering of the circles for each smoothing D_ε and associate the i -th tensor factor of M_ε to the i -th circle of D_ε . Take elements ε and $\varepsilon' \in \{0, 1\}^n$ such that $\varepsilon_j = 0$ and $\varepsilon'_j = 1$ for some j and that $\varepsilon_i = \varepsilon'_i$ for any $i \neq j$. For such a pair $(\varepsilon, \varepsilon')$, we will define a map $d_{\varepsilon \rightarrow \varepsilon'}: M_\varepsilon \rightarrow M_{\varepsilon'}$ as follows.

In the case where two circles of D_ε merge into one circle of $D_{\varepsilon'}$, the map $d_{\varepsilon \rightarrow \varepsilon'}$ is the identity on all factors except the tensor factors corresponding to the merged circles where it is a multiplication map $m: V \otimes V \rightarrow V$ given by:

$$m(1 \otimes 1) = 1, m(1 \otimes X) = m(X \otimes 1) = X, m(X \otimes X) = 0.$$

In the case where one circle of D_ε splits into two circles of $D_{\varepsilon'}$, the map $d_{\varepsilon \rightarrow \varepsilon'}$ is the identity on all factors except the tensor factor corresponding to the split circle where it is a comultiplication map $\Delta: V \rightarrow V \otimes V$ given by:

$$\Delta(1) = 1 \otimes X + X \otimes 1, \Delta(X) = X \otimes X.$$

If there exist distinct integers i and j such that $\varepsilon_i \neq \varepsilon'_i$ and that $\varepsilon_j \neq \varepsilon'_j$, then define $d_{\varepsilon \rightarrow \varepsilon'} = 0$.

In this setting, we define a map $d^i: C^i(D) \rightarrow C^{i+1}(D)$ by $\sum_{|\varepsilon|=i} d_\varepsilon^i$, where $d_\varepsilon^i: M_\varepsilon \rightarrow C^{i+1}(D)$ is defined by

$$d^i(v) := \sum_{|\varepsilon'|=i+1} (-1)^{l(\varepsilon, \varepsilon')} d_{\varepsilon \rightarrow \varepsilon'}(v).$$

Here $v \in M_\varepsilon \subset C^i(D)$ and $l(\varepsilon, \varepsilon')$ is the number of 1's in front of (in our order) the factor of ε which is different from ε' .

We can check that $(C^i(D), d^i)$ is a cochain complex and we denote its i -th homology group by $H^i(D)$. We call these the unnormalized homology groups of D . Since the map d^i preserves the grading of $C^i(D)$, the group $H^i(D)$ has a graded structure $H^i(D) = \bigoplus_{j \in \mathbf{Z}} H^{i,j}(D)$ induced by that of $C^i(D)$. For any link diagram D , we define its Khovanov homology $\text{KH}^{i,j}(D)$ by

$$\text{KH}^{i,j}(D) = H^{i+n_-, j-n_++2n_-}(D),$$

where n_+ and n_- are the number of the positive and negative crossings of D , respectively. The grading i is called the homological degree and j is called the q -grading.

Theorem 2.1 ([1], [3]). *For any oriented link L and a diagram D of L , the homology group $\text{KH}(D)$ is preserved under the Reidemeister moves. In this sense, we can denote $\text{KH}(L) = \text{KH}(D)$. Moreover the graded Euler characteristic of the homology $\text{KH}(L)$ equals the Jones polynomial of L , that is,*

$$V_L(t) = (q + q^{-1})^{-1} \sum_{i,j \in \mathbf{Z}} (-1)^i q^j \text{rank KH}^{i,j}(L) \Big|_{q=-t^{\frac{1}{2}}},$$

where $V_L(t)$ is the Jones polynomial of L .

By the definition, we have the following.

Proposition 2.2. *For any oriented link L , we have*

$$\max\{i \in \mathbf{Z} \mid \text{KH}^i(L) \neq 0\} \leq c_+(L),$$

where $c_+(L)$ is defined in Theorem 1.2.

Proof. For any diagram D of L , by the definition, we have $H^i(D) = 0$ if $i > c(D)$, where $c(D)$ is the number of the crossings of D . Hence $\mathrm{KH}^i(L) = \mathrm{KH}^i(D) = 0$ for $i > c_+(D)$. Since $\mathrm{KH}^i(L)$ does not depend on the choice of D , we have $\mathrm{KH}^i(L) = 0$ for $i > c_+(K)$. \square

2.2. Example. For example, the Khovanov homology of the left-handed trefoil knot K (depicted in Figure 4) is given as follows.

$$\mathrm{KH}^{i,j}(K) = \begin{cases} \mathbf{Q} & \text{if } (i,j) = (0,-1), (0,-3), (-2,-5), (-3,-9), \\ 0 & \text{otherwise.} \end{cases}$$

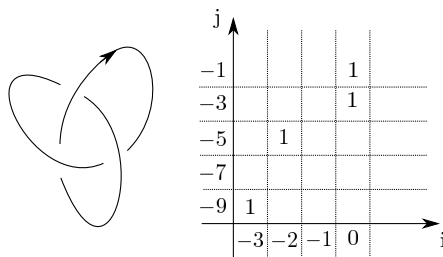


FIGURE 4. The left-handed trefoil and the table of the Khovanov homology of the knot whose (i, j) element is $\dim_{\mathbb{Q}} \mathrm{KH}^{i,j}(K)$.

2.3. Main tool. Our main tool is the following theorem proved by Wehrli [6] and Champanerkar and Kofman [2]. The **0-smoothing** of a diagram D is the disjoint circles obtained from D by 0-smoothing all crossings (see Figure 3). Similarly, we define the **1-smoothing** of a diagram D . Then we have the following.

Theorem 2.3 ([6], [2]). *Let D be a link diagram. If $H^{i,j}(D) \neq 0$, we have*

$$s_1(D) - 2 - c(D) \leq j - 2i \leq 2 - s_0(D),$$

where $c(D)$ is the number of the crossings of D , and $s_0(D)$ and $s_1(D)$ are the numbers of the circles appearing in the $\mathbf{0}$ -smoothing and the $\mathbf{1}$ -smoothing of D , respectively.

3. THE MAIN THEOREM AND ITS PROOF

In this section, we show our main theorem (Theorem 1.3). To prove this theorem, we first compute the maximal degree of the Jones polynomial and prove that it is proportional to the number n of twists if n is sufficiently large (Lemma 3.1). From this fact, we can relate the Khovanov homology with the number of twists since the Jones polynomial is the graded Euler characteristic of the Khovanov homology.

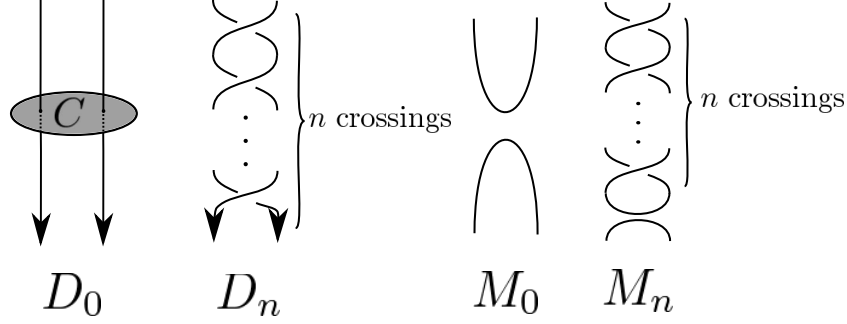
Let L_0 be an oriented link and D_0 be a diagram of L_0 . Let C be a disk as in Figure 5 and D_n be the diagram obtained from D_0 by adding n half twists at C as in Figure 5. The diagram D_n is a diagram of $t_{n/2}(L_0; C)$. Put $L_n := t_{n/2}(L_0; C)$.

To prove Theorem 1.3, we use the following lemma.

Lemma 3.1. *There is a positive integer N such that for any $n \geq N$*

$$\maxdeg V_{L_{n+1}}(t) - \maxdeg V_{L_n}(t) = \frac{3}{2},$$

where $\max \deg V_L(t) := \max\{i \in \mathbf{Z}[\frac{1}{2}] \mid \text{the coefficient of } t^i \text{ in } V_L(t) \text{ is not zero}\}$.

FIGURE 5. The diagrams D_0 , D_n , M_0 and M_n .

Proof. The Kauffman bracket of an (unoriented) link diagram is given as follows:

- $\langle \nearrow \searrow \rangle = A \langle \searrow \rangle \langle \nearrow \rangle + A^{-1} \langle \nearrow \searrow \rangle$,
- $\langle \bigcirc \rangle = 1$,
- $\langle D \sqcup \bigcirc \rangle = (-A^{-2} - A^2) \langle D \rangle$.

For any link L , the Jones polynomial is given by

$$V_L(t) = (-A)^{-3w(D)} \langle D \rangle \Big|_{A=t^{-\frac{1}{4}}},$$

where D is a diagram of L and $w(D)$ is the writhe of D . Hence we have

$$\begin{aligned}
 (3.1) \quad V_{L_n}(t) &= (-A)^{-3w(D_n)} (A \langle D_{n-1} \rangle + A^{-1} \langle M_{n-1} \rangle) \Big|_{A=t^{-\frac{1}{4}}} \\
 &= (-A)^{-3w(D_{n-1})} (-A)^{-3} A \langle D_{n-1} \rangle \Big|_{A=t^{-\frac{1}{4}}} \\
 &\quad + (-A)^{-3w(D_n)} A^{-1} (-A)^{-3(n-1)} \langle M_0 \rangle \Big|_{A=t^{-\frac{1}{4}}} \\
 &= -t^{\frac{1}{2}} V_{L_{n-1}}(t) + (-1)^{-3(w(D_n)+n-1)} A^{-3(w(D_0)+n)-3n+2} \langle M_0 \rangle \Big|_{A=t^{-\frac{1}{4}}} \\
 &= -t^{\frac{1}{2}} V_{L_{n-1}}(t) + (-1)^{-3(w(D_n)+n-1)} t^{\frac{3}{2}n} (A^{-3w(D_0)+2} \langle M_0 \rangle) \Big|_{A=t^{-\frac{1}{4}}},
 \end{aligned}$$

where M_n is the diagram depicted in Figure 5.

For contradiction, assume that for any positive integer n , $\maxdeg V_{L_n}(t) - \maxdeg V_{L_{n-1}}(t) \leq \frac{1}{2}$. Then we have $\maxdeg V_{L_n}(t) \leq \frac{1}{2}n + \maxdeg V_{L_0}(t)$. Hence, from (3.1), there is a positive integer N such that

$$\maxdeg V_{L_n}(t) = \maxdeg(t^{\frac{3}{2}n} (A^{-3(w(D_0)+2)} \langle M_0 \rangle) \Big|_{A=t^{-\frac{1}{4}}}(t))$$

for any $n \geq N$. In particular, $\maxdeg V_{L_{N+1}}(t) - \maxdeg V_{L_N}(t) = \frac{3}{2}$. This is a contradiction. Hence, there is a positive integer N such that

$$\maxdeg V_{L_N}(t) - \maxdeg V_{L_{N-1}}(t) > \frac{1}{2}.$$

By the skein relation $t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{1/2} - t^{-1/2})V_{L_0}(t)$ for a usual skein triple (L_+, L_-, L_0) , we obtain

$$V_{L_{N+1}}(t) = t^2 V_{L_{N-1}}(t) + (t^{\frac{3}{2}} - t^{\frac{1}{2}}) V_{L_N}(t).$$

Hence, we have

$$\maxdeg V_{L_{N+1}}(t) = \frac{3}{2} + \maxdeg V_{L_N}(t).$$

By repeating this process, we have $\maxdeg V_{L_{n+1}}(t) = \frac{3}{2} + \maxdeg V_{L_n}(t)$ for any $n \geq N$. \square

Proof of Theorem 1.3. Let L_0 , L_n , D_0 and D_n be as above. Note that $s_0(D_n) = s_0(D_0)$, where $s_0(D_0)$ and $s_0(D_n)$ are introduced in Theorem 2.3. By Theorem 2.3, if $H^{i,j}(D_n) \neq 0$, we have

$$(3.2) \quad j - 2i \leq 2 - s_0(D_0).$$

Since the graded Euler characteristic of the Khovanov homology is the Jones polynomial (Theorem 2.1), we obtain

$$\max\{j \in \mathbf{Z} \mid \mathrm{KH}^{*,j}(D) \neq 0\} \geq 2 \maxdeg V_D(t) + 1$$

for any link diagram D . Moreover

$$(3.3) \quad \max\{j \in \mathbf{Z} \mid H^{*,j}(D) \neq 0\} \geq 2 \maxdeg V_D(t) + 1 - c_+(D) + 2c_-(D),$$

where $c_+(D)$ and $c_-(D)$ are the numbers of the positive and negative crossings of D , respectively. Put $f(D_n) := 2 \maxdeg V_{L_n}(t) + 1 - c_+(D_n) + 2c_-(D_n)$ and $k(D_n) := \frac{1}{2}(f(D_n) - 2 + s_0(D_0))$. From (3.2), (3.3), the definition of H^i (the unnormalized Khovanov homology) and Figure 6, we have

$$(3.4) \quad k(D_n) \leq \max\{i \in \mathbf{Z} \mid H^i(D_n) \neq 0\} \leq c(D_n) = c(D_0) + n.$$

By Lemma 3.1, there is a positive integer N such that $k(D_{n+1}) - k(D_n) = 1$ for any $n \geq N$. In particular $k(D_{N+1}) = n - N + k(D_{N+1})$. From (3.4), we have

$$n - N + k(D_{N+1}) \leq \max\{i \in \mathbf{Z} \mid \mathrm{KH}^i(L_n) \neq 0\} + c_-(D_0) \leq c(D_0) + n.$$

This implies Theorem 1.3. \square

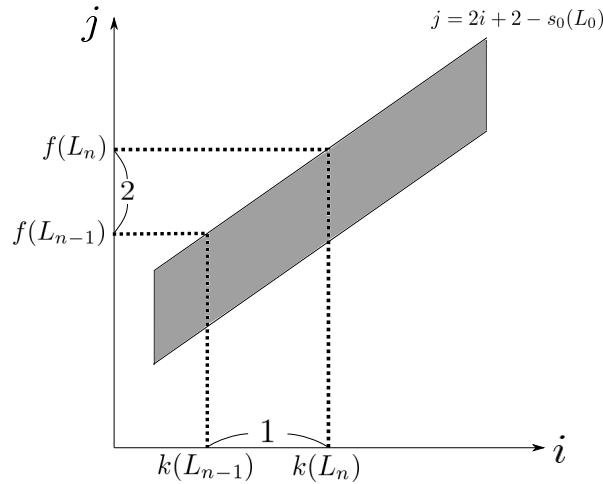


FIGURE 6. The gray parallelogram contains the support of rank $H^{i,j}$.

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